FRACTIONAL QUANTUM HALL HIERARCHY

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Motivation

i) Many Body Theory of Solids

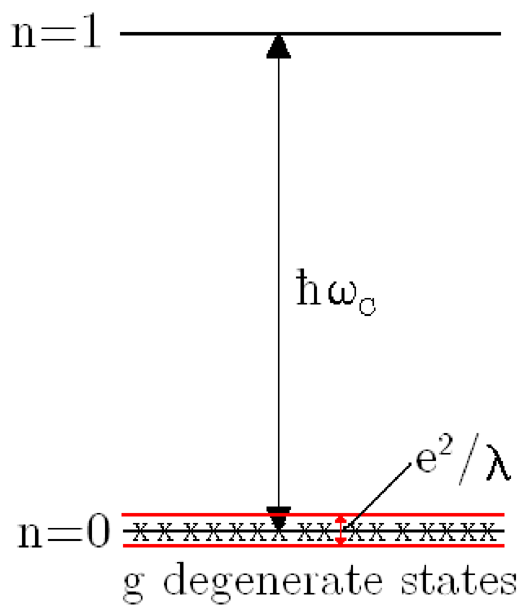
1950s- **H_0 SINGLE PARTICLE ENERGIES**
H_I MANY BODY PERTURBATION THEORY

Landau Theory, Gell Mann – Bruckner, Hubbard

Electron Self Energy: $\Sigma = \Sigma_1 + i\Sigma_2$

QPs: $\varepsilon_{QP} = \varepsilon_0 + \Sigma, \qquad V_{QP}(\vec{p} - \vec{p}')$

ii) Strongly Interacting Systems



H_I too strong to treat as perturbation

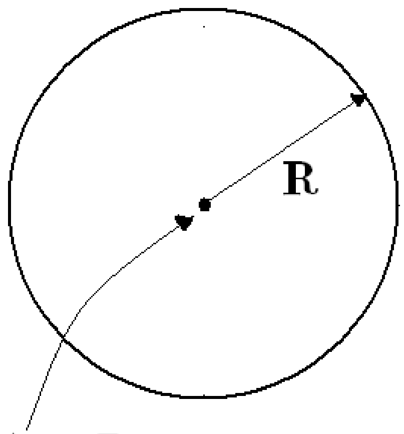
FQH Effect – a PARADIGM

N electrons in $g = 2\ell + 1$ degenerate states

$\hbar\omega_C >> e^2/\lambda$

No kinetic energy, $H_I = V_{ee}$ only energy scale

Haldane spherical geometry



$$4\pi R^2 B = 2Q\left(hc/e\right)$$

$$R^2 = Q\lambda^2$$

$$\lambda^2 = \hbar c/(eB)$$

SINGLE PARTICLE STATES (Monopole Harmonics)

$|Q,\ell,m\rangle$ eigenstates of $\begin{cases} \hat{\ell}^2 \\ \hat{\ell}_z \\ \hat{H}_0 \end{cases}$

$$E_{Q\ell m} = \frac{\hbar\omega_C}{2Q}\left[\ell\left(\ell+1\right) - Q^2\right]$$

$E \geq 0 \Rightarrow \ell_n = Q + n,$ n=0,1,2,… angular momentum shells

$g_n = 2\ell_n + 1$

Lowest LL $\ell_0 = Q$

Filling factor LLL $\nu = \dfrac{N}{g_0} \Rightarrow N\left(\#\text{flux quanta through sample}\right)^{-1}$

Numerical Studies

N –Fermion States

$$\left|m_1, m_2 \ldots m_N\right\rangle = c_N^\dagger c_{N-1}^\dagger \ldots c_1^\dagger \left|0\right\rangle, \quad m_i \in g_0$$

$$H_I = \sum_{\langle i,j\rangle} \frac{e^2}{r_{ij}}$$

Diagonalize H_I

$\left\langle m_1' m_1' \ldots m_N' \right| H_I \left| m_1 m_1 \ldots m_N \right\rangle$ vanishes unless

i) $M = \sum_i m_i = \sum_i m_i'$

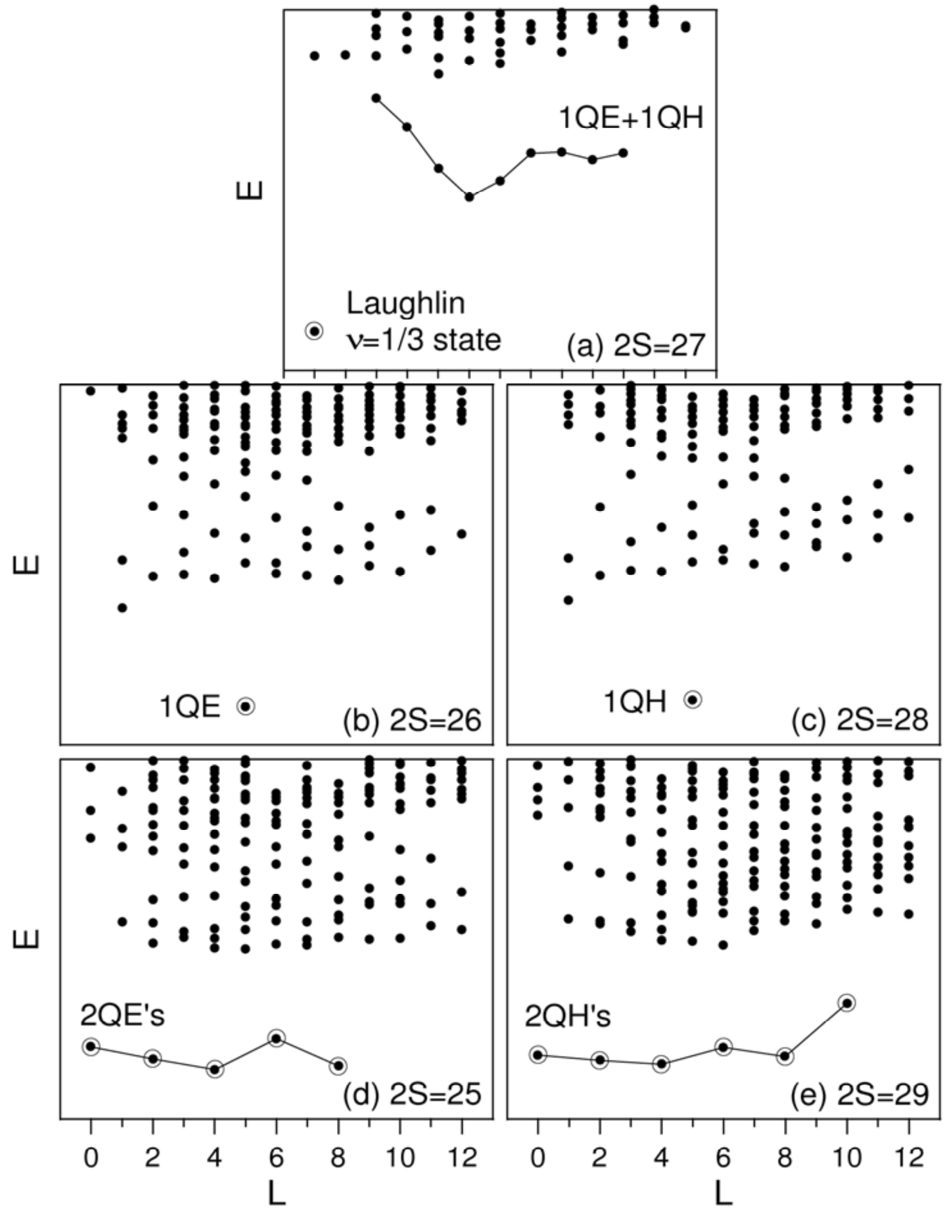
ii) $\left|m_1 m_2 \ldots m_N\right\rangle$ and $\left|m_1' m_1' \ldots m_N'\right\rangle$ differ by no more than two members

Wigner – Eckart theorem

$$\left\langle L'M'\alpha' \right| H_I \left| LM\alpha \right\rangle = \delta_{LL'} \delta_{MM'} \left\langle L\alpha' \right| H_I \left| L\alpha \right\rangle$$

(independent of M)

Numerical Studies



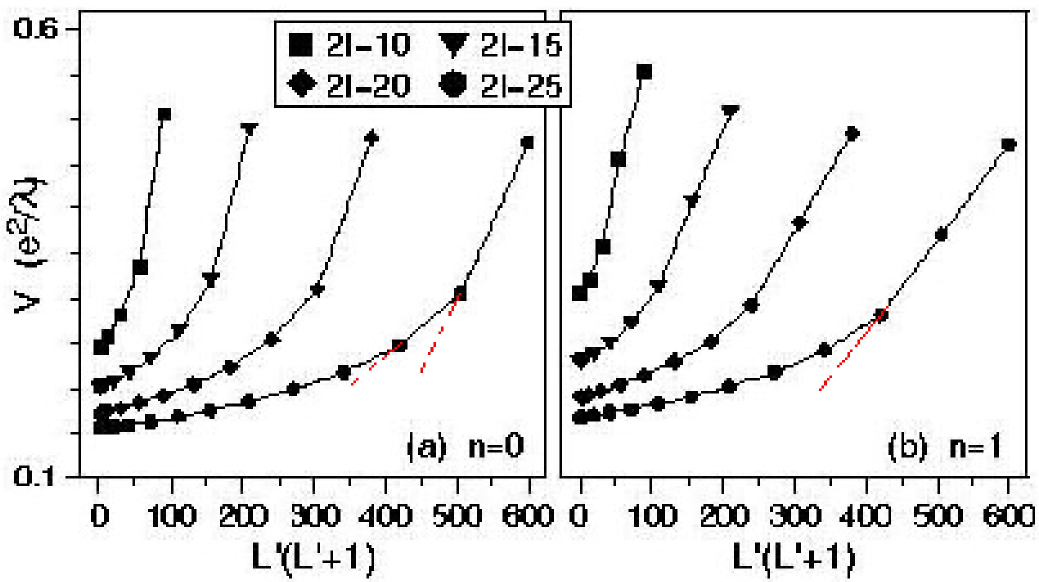
The energy spectra of 10 electrons in the lowest Landau level calculated on a Haldane sphere with $2Q$ between 25 and 29. The open circles and solid lines mark the lowest-energy bands with the fewest composite fermion quasiparticles.

Pseudopotentials

$$E_\alpha(L) = \frac{1}{2}N(N-1)\sum_{L_{12}} P_{L\alpha}(L_{12})V(L_{12})$$

$$L_{12} = 2\ell - R, \qquad R = 1,3,5\ldots$$

PSEUDOPOTENTIAL $V(L_{12}) = \left\langle \Psi_{L_{12}}(r_{12}) \middle| \frac{e^2}{r_{12}} \middle| \Psi_{L_{12}}(r_{12}) \right\rangle$



Pseudopotential *V(L')* of the Coulomb interaction in the lowest (a) and the first-excited Landau level (b) as a function of squared pair angular momentum *L'(L'+1).* Squares (*l=5*), triangles (*l=15/2*), diamonds (*l=10*), and circles (*l=25/2*) indicate data for different values of *Q=l*.

Fractional Parentage

$$\left|\ell^N, L\alpha\right\rangle = \sum_{L_{12}, L'\alpha'} G_{L\alpha, L'\alpha'}(L_{12}) \left|\ell^2, L_{12}; \ell^{N-2}, L'\alpha'; L\right\rangle \qquad \text{(A)}$$

$$P_{L\alpha}(L_{12}) = \sum_{L'\alpha'} \left|G_{L\alpha, L'\alpha'}(L_{12})\right|^2$$

USEFUL IDENTITY

If $\hat{L} = \sum_i \hat{\ell}_i, \qquad \hat{L}_{ij} = \hat{\ell}_i + \hat{\ell}_j$

$$\hat{L}^2 + N(N-2)\hat{\ell}^2 - \sum_{\langle ij \rangle} \hat{L}_{ij}^2 = 0 \qquad \text{(B)}$$

Take expectation values of (B) using wavefunction (A)

THEOREMS

1. $$\sum_{L'} P_{L\alpha}(L') = 1$$

2. $$\frac{1}{2} N(N-1) \sum_{L'} P_{L\alpha}(L') L'(L'+1) = L(L+1) + N(N-2)\ell(\ell+1)$$

Harmonic Pseudopotentials

$$V_H\left(\hat{L}'\right) = A + B\hat{L}'^2$$

For $V\left(\hat{L}'\right) = V_H$

$$E_\alpha = N\left[\frac{1}{2}(N-1)A + B(N-2)\ell(\ell+1)\right] + BL(L+1)$$

Note: i) ***$E_\alpha(L)$* is independent of α for *V_H***
ii) ***$E(L)$* increases as *$L(L+1)$***

V_H DOES NOT REMOVE THE DEGENERACY OF DIFFERENT MULTIPLETS WITH THE SAME L.

CORRELATIONS: **ONLY $\Delta V\left(L'\right)$ CAUSES CORREALTIONS**

$\Delta V = V(L') - V_H(L')$ **ANHARMONIC PART OF *V(L')***

Relative pair angular momentum $R = 2\ell - L' = 1,3,5\ldots$

We often write ***V(R)*** which stands for $V\left(2\ell - R\right)$

Simplest Anharmonic Pseudopotential

$$\Delta V(R) = k\delta(R,1)$$

i) $k > 0 \Rightarrow$ Lowest multiplet for each L has $P_{L\alpha}(R=1)$ a minimum.

ii) $k < 0 \Rightarrow$ Lowest multiplet for each L has $P_{L\alpha}(R=1)$ a maximum.

Case i) is exactly what is meant by "Laughlin correlations"

It is why Laughlin wave function is the exact solution to a very short range potential

Case ii) electrons have tendency to form pairs instead of Laughlin correlations

Jain's Composite Fermion Picture

i) **Make Chern-Simons transformation by adding to each electron a flux tube giving a CS magnetic field**

$$b(\vec{r}) = -\gamma\phi_0 \sum_i \delta(\vec{r} - \vec{r}_i) \qquad \phi_0 = \frac{hc}{e}, \quad \gamma \text{ is a constant}$$

This CS magnetic field, added via a gauge transformation, has no effect on classical equation of motion.

ii) **Apply mean field approximation to electron charge and flux, and set $\gamma = 2p_0$ to form composite Fermions**. This gives an <u>effective CF filling</u>

$$\frac{1}{\nu_0^*} = \frac{1}{\nu_0} - 2p_0$$

If $\nu_0^* = n_0$, an integer $\Longrightarrow$ Jain sequence of filled CF levels

$$\nu = \begin{cases} \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots \\ 1, \frac{2}{3}, \frac{2}{5}, \ldots \end{cases}$$

Effective monopole strength

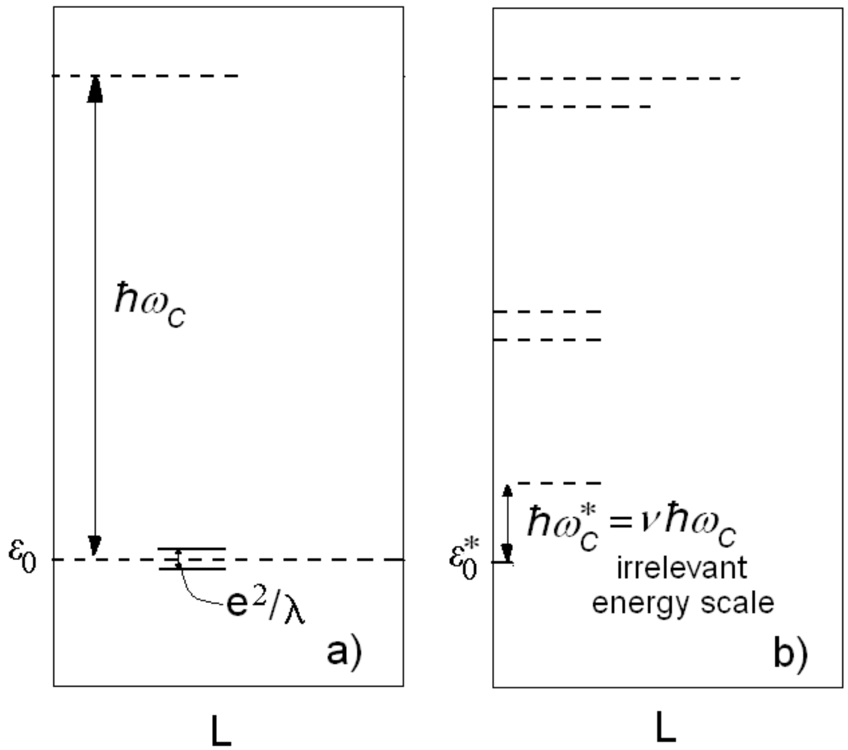
$$2Q^* = 2Q - 2p_0(N-1)$$

Q^* is the angular momentum of CF shell

The effective CF monopole strength $2Q^*$, the number of CF quasiparticles (quasiholes n_{QH} and quasielectrons n_{QE}), the quasiparticle angular momentum ℓ_{QH} and ℓ_{QE}, and the angular momenta L of the lowest lying band of multiplets for ten-electron system at 2Q between 29 and 21.

$2Q$	29	28	27	26	25	24	23	22	21
$2Q^*$	11	10	9	8	7	6	5	4	3
n_{QH}	2	1	0	0	0	0	0	0	0
n_{QE}	0	0	0	1	2	3	4	5	6
ℓ_{QH}	5.5	5	4.5	4	3.5	3	2.5	2	1.5
ℓ_{QE}	6.5	6	5.5	5	4.5	4	3.5	3	2.5
L	0,2,4,6,8,10	5	0	5	0,2,4,6,8	$1, 3^2, 4, 5, 6, 7, 9$	$0, 2^2, 4^2, 5, 6, 8$	1,3,5	0

Why Does Composite Fermion Picture Work?



b) must reduce to a) when $e^2/(\lambda\hbar\omega_C) \to 0$

Scale $\hbar\omega_C^*$ is irrelevant for $\hbar\omega_C >> e^2/\lambda$

CS gauge interaction $\propto \hbar\omega_C^* \propto B$

Coulomb interaction $\propto e^2/\lambda \propto \sqrt{B}$

CANCELATION IMPOSSIBLE !

Adiabatic Addition of CS Flux

Relative coordinate WF of pair (on a plane):

$$\Psi_{nm}(r,\phi) = e^{im\phi} u_{nm}(r); \qquad r = |r_1 - r_2|, \phi = \phi_1 - \phi_2$$

1. Add -γ CS flux quanta via gauge transformation:

$$\Psi_{nm} \to e^{i(m-\gamma)\phi} u_{nm}(r)$$

Changes only phase of WF. If γ is not an even integer it changes statistics.

2. Add –γ CS flux adiabatically:

$$\Psi_{nm} \to e^{im\phi} u_{n,m+\gamma}(r)$$

No phase change, radial function goes from u_{nm} to $u_{n,m+\gamma}$.

For $\gamma = 2 \Rightarrow$ Orbit has changed to new "Laughlin correlated" pair orbit

Laughlin correlations without need of M.F. approx and irrelevant energy scale.

No change in ε_{nm} without H_I.

Laughlin WF for pair:

$$\Psi(\vec{r}_1,\vec{r}_2) = \Psi^{REL}_{0m}(\vec{r}_{12})\Psi^{CM}_{00}(\vec{R}_{12}) \to (z_1 - z_2)^{|m|} \exp\left(-\frac{r_1^2 + r_2^2}{4\lambda^2}\right)$$

Composite Fermion Hierarchy

$$\nu_0^{-1} = 2p_0 + \nu_0^{*-1}$$

$\nu_0^* = n_0$ gives JAIN SEQUENCE

i) Suppose $\nu_0^* \neq$ integer

Write $\nu_0^* = n_0 + \nu_1$

ν_1= QP filling of CF shell

ii) Assume (like Haldane) QPs are L.C. $\dfrac{1}{\nu_1} = 2p + \dfrac{1}{\nu_1^*}$

$\nu_1^* = n_1$ gives DAUGHTER IQL STATE

iii) In general

$$\frac{1}{\nu_j} = 2p_j + \frac{1}{n_j + \nu_{j+1}}$$

Series terminates as soon as: ν_{j+1}= 0

CF hierarchy gives:

1) Jain sequence of CF integrally filled levels when n_0 is integer
2) Haldane hierarchy of all odd denominator fraction (in terms of CF QPs) when QPs are Laughlin correlated at each level.

CF Hierarchy Doesn't Agree With "Numerical Experiment"

$N = 8, \quad \ell_0 = 18$

$2\ell_1 = 2\ell_0 - 2(N-1) = 4$

$\ell_1 = 2$ holds 5 CFs leaving N_{QE}=3 in shell $\ell_{QE} = 3$

3 QEs with $\ell_{QE} = 3$ gives a low lying band $L = 0 \oplus 2 \oplus 3 \oplus 4 \oplus 6$

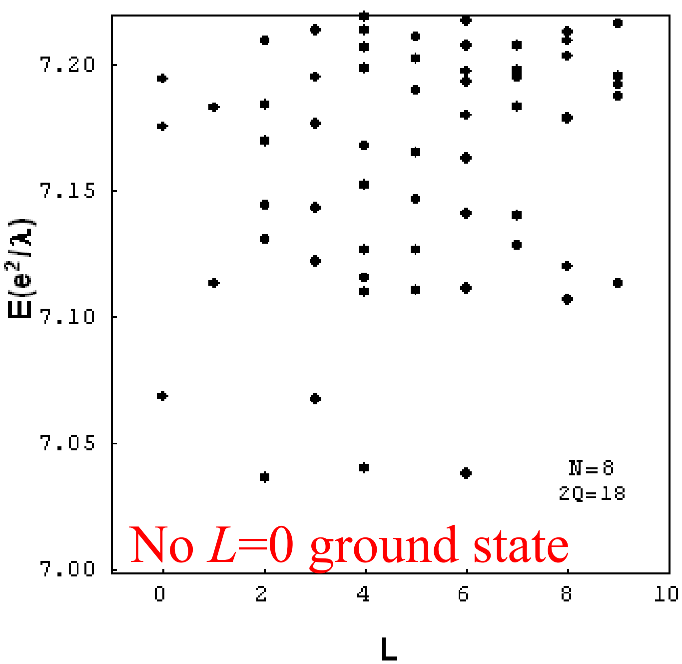
A new CF transformation will lead to 3 CFQE in $\ell^*_{QE} = 1$

This gives a daughter state $L = 0$

$$\frac{1}{\nu_1} = 2p_1 + \frac{1}{n_2 + \nu_2} = 2 + 1 = 3$$

$$\frac{1}{\nu_0} = 2p_0 + \frac{1}{n_1 + \nu_1} = 2 + \frac{1}{(1 + 1/3)} = \frac{11}{4}$$

$$\therefore \nu_0 = 4/11$$

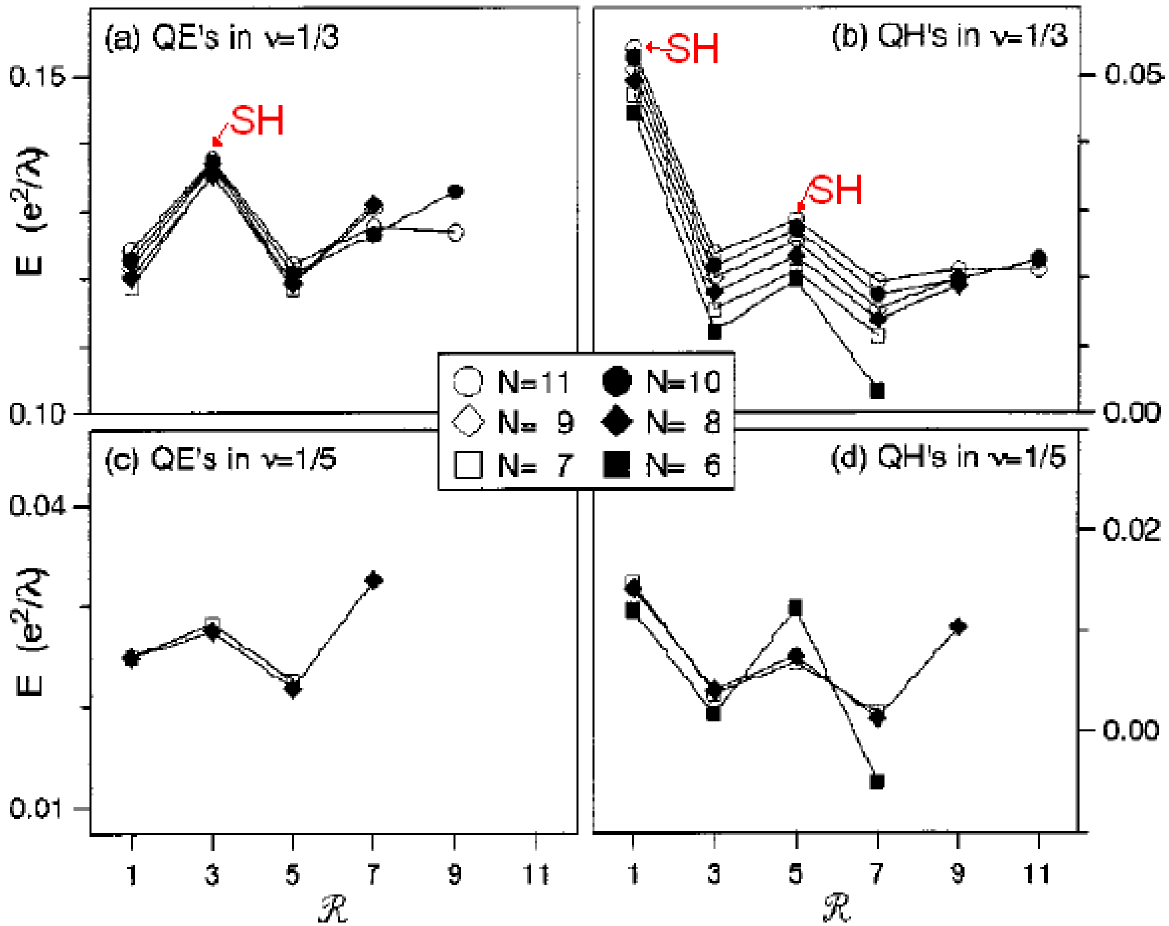


Why Does CF Hierarchy Fail?

It assumes Laughlin correlations of QPs, but $V_{QE}(R)$ is not "superharmonic" at R=1 and $V_{QH}(R)$ is not "superharmonic" at R=3.

No $\nu_{QE} = 1/3$ state of spin polarized system

No $\nu_{QH} = 1/5$ state of spin polarized system



The pseudopotentials of pairs of quasielectrons (left) and quasiholes (right) in Laughlin $\nu = 1/3$ (top) $\nu = 1/5$ and (bottom), as a function of pair relative angular momentum *R.*

Pairing and larger clusters

Because $V_{QE}(R=1)$ won't support LCs but will tend to have pairs formed, let's

i) assume N electrons form $N_P=N/2$ pairs

ii) $\ell_P = 2\ell - 1$

We treat pairs as Fermions. We prevent violation of Pauli principle by restricting pair angular momentum of N_P pairs to values $< 2\ell_{FP}$ **where**

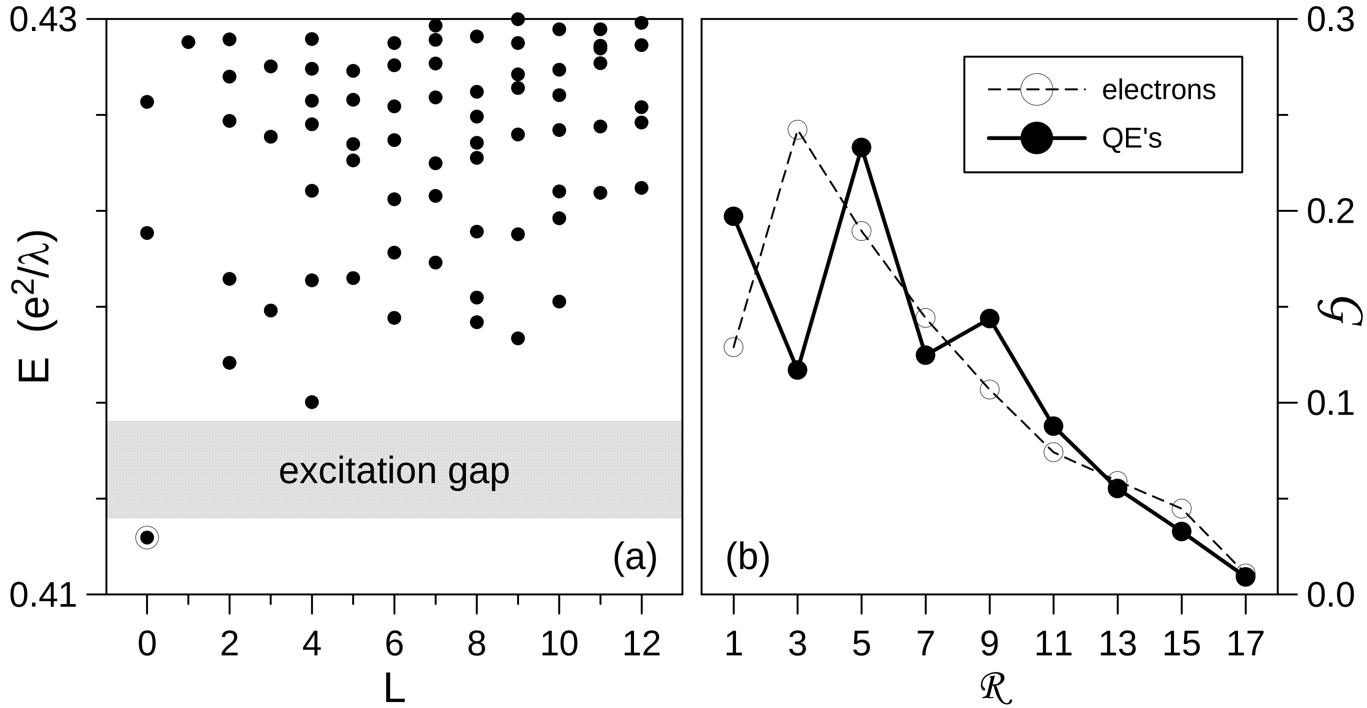
$$2\ell_{FP} = 2\ell_P - 3\left(N_P - 1\right) \qquad \text{(CS transformation)}$$

Last term is chosen so that $\nu_{FP} = 1$ when $\nu = 1$. This gives $\frac{1}{\upsilon_{FP}} = \frac{4}{\upsilon_{QE}} - 3$

The 4 comes from $\ell_P = 2\ell - 1$ and $N_P = N/2$. This leads to IQL states of pairs at:

ν_{FP}	1/3	1/5	1/7	1/9
ν	2/3	1/2	2/5	1/3
ν_{QE}	5/13	3/8	7/19	4/11
ν	2/7	1/4	2/9	1/5
ν_{QH}	5/17	3/10	7/23	4/13

Quasielectron energy spectra



$$N_{QE} = 10 \qquad 2\ell_{QE} = 17$$

$$\nu_{QE} = 1/2 \qquad \nu^{-1} = 2 + \frac{1}{1 + 1/2} = \frac{8}{3}$$

Unsolved questions

1. $\nu_{\mathrm{QE}} = 1/3$ occurs at $2\ell = 3N-7$ not at $2\ell = 3N-5$ (expected for pairs) nor $2\ell = 3N-3$ (for LC electrons).

2. $\nu_{\mathrm{QE}} = 1/2$ occurs at $2\ell = 2N-3$ or its conjugate $2\ell = 2N+1$ but only when smaller of N_{QE} or N_{QH} is a multiple of 4.

We don't understand why. We clearly are missing something. But we are certain that there is no 2nd generation of CFs at $\nu_{\mathrm{QE}} = 1/3$ or $\nu_{\mathrm{QH}} = 1/5$ for a totally spin polarized system.